Available online at www.sciencedirect.com
ScienceDirect
Topology 45 (2006) 919–927
TOPOLOGY
www.elsevier.com/locate/top

A fake projective plane with an order 7 automorphism

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Abstract

A fake projective plane is a compact complex surface (a compact complex manifold of dimension 2) with the same Betti numbers as the complex projective plane, but not isomorphic to the complex projective plane. As was shown by Mumford, there exists at least one such surface.

In this paper we prove the existence of a fake projective plane which is birational to a cyclic cover of degree 7 of a Dolgachev surface.

MSC: 14J29; 14J27

Keywords: Fake projective plane; Dolgachev surface; Cyclic covering

1. Introduction

It is known that a compact complex surface with the same Betti numbers as the complex projective plane $\mathbb{P}^2$ is projective (see e.g. [1]). Such a surface is called *a fake projective plane* if it is not isomorphic to $\mathbb{P}^2$.

Let S be a fake projective plane, i.e. a surface with $b_1(S) = 0, b_2(S) = 1$ and $S \ncong \mathbb{P}^2$. Then its canonical bundle is ample and S is of general type. So a fake projective plane is nothing but a surface of general type with $p_g = 0$ and $c_1(S)^2 = 3c_2(S) = 9$. Furthermore, its fundamental group $\pi_1(S)$ is infinite. Indeed, by Castelnuovo's rationality criterion, its second plurigenus $P_2(S)$ must be positive and hence the first Chern class $-c_1(S)$ of the cotangent bundle of S can be represented by a Kähler form. Then it follows from the solution of Yau [10] to the Calabi conjecture that S admits a Kähler–Einstein metric, and hence its universal cover is the unit ball in $\mathbb{C}^2$.

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0040-9383/$ - see front matter
doi:10.1016/j.top.2006.06.006

Mumford [8] first discovered a fake projective plane. His construction uses the theory of the p-adic unit ball by Kurihara [6] and Mustafin [9]. Later, using the same idea, Ishida and Kato [3] proved the existence of at least two more. It is not known whether any of these surfaces admits an order 7 automorphism.

In this paper we prove the existence of a fake projective plane with an order 7 automorphism (see Theorem 3.1). Our construction uses Ishida's description [2] of an elliptic surface Y covered by a (blow-up) of Mumford's surface M. Recall that there exist an unramified Galois cover $V \to M$ of degree 8, and a simple group G of order 168 acting on V such that Y is the minimal resolution of the quotient V/G, and M is the quotient of V by a 2-Sylow subgroup of G. Our surface is birationally isomorphic to a cyclic cover of degree 7 of a cyclic cover of degree 3 of Y. Mumford's surface M is a degree 21 non-Galois cover of Y [2], but it is not clear whether it is different from our surface.

The elliptic surface Y has not up to now been constructed directly (although its properties are stated explicitly), so it does not yet yield an alternative approach to Mumford's surface. Thus we do not know how to construct our surface in a direct way.

2. Two Dolgachev surfaces

In [2] Ishida discusses an elliptic surface Y with $p_g = q = 0$ having two multiple fibres of multiplicity 2 and 3 respectively, and proves that the Mumford fake plane M is a cover of Y of degree 21, but not a Galois cover.

The surface Y is a Dolgachev surface [1]. In particular, it is simply connected and of Kodaira dimension 1. Besides the two multiple fibres, its elliptic fibration $|F_Y|$ has four more singular fibres F_1, F_2, F_3, F_4, all of type I_3. It has also a sextuple section E which is a smooth rational curve meeting one component of each of F_1, F_2, F_3 in six points, and two components of F_4 in one point and five points each. Write

$$F_i = A_{i1} + A_{i2} + A_{i3}, \quad i = 1, 2, 3, 4.$$

After suitable renumbering, one may assume that

$$E \cdot F_Y = E \cdot A_{13} = E \cdot A_{23} = E \cdot A_{33} = 6, \quad E \cdot A_{41} = 1, \quad E \cdot A_{43} = 5.$$

Note that $E^2 = -3$. The six curves

$$A_{11}, A_{12}, A_{21}, A_{22}, A_{31}, A_{32}$$

form a Dynkin diagram of type $A_2^{\oplus 3}$,

$$(-2) — (-2) \quad (-2) — (-2) \quad (-2) — (-2)$$

and hence can be contracted to three singular points of type $\frac{1}{3}(1, 2)$. Let

$$\sigma : Y \to Y'$$

be the contraction morphism. Since Y is simply connected, $H^2(Y, \mathbb{Z})$ has no torsion and is a lattice under intersection pairing. Let

$$R \subset H^2(Y, \mathbb{Z})$$

be the sublattice generated by the classes of the six exceptional curves, and let $\overline{R}$ and $R^{\perp}$ be its primitive closure and its orthogonal complement in $H^2(Y, \mathbb{Z})$, respectively. Since $H^2(Y, \mathbb{Z})$ is unimodular of rank 10, we see that

$$\operatorname{rank} R^{\perp} = 4, \quad \operatorname{disc}(\overline{R}) \cong -\operatorname{disc}(R^{\perp}).$$

Here, we use the following notation: for an even lattice L, $\operatorname{disc}(L)$ denotes the discriminant group of L, that is, the quotient group L^*/L of the dual $L^* := \operatorname{Hom}(L, \mathbb{Z})$ by L together with a $\mathbb{Q}/2\mathbb{Z}$-valued quadratic form induced by the integral quadratic form of L. Its length $l(\operatorname{disc}(L))$ is the minimum number of generators of the finite abelian group L^*/L.
Note that the classes of the three curves

$$E, \quad A_{41}, \quad A_{42}$$

generates a rank 3 sublattice of $R^{\perp}$ whose discriminant group has length 1 (a cyclic group of order 7), and hence

$$l(\operatorname{disc}(\overline{R})) = l(\operatorname{disc}(R^{\perp})) \leq 2.$$

Since $\operatorname{disc}(R)$ is 3-elementary of length 3, we see that $\overline{R}$ is an overlattice of index 3 of R. This implies that there is a cyclic cover of degree 3

$$X \to Y'$$

branched exactly at the three singular points of Y'. Then X is a nonsingular surface. It turns out that X is another Dolgachev surface. More precisely, we have

Proposition 2.1. *The surface X is a minimal elliptic surface of Kodaira dimension* 1 *with $p_g = q = 0$ with one fibre of multiplicity* 2*, one of multiplicity* 3*, one singular fibre of type I_9 and three of type I_1. Furthermore, it has* 3 *sextuple sections E_1, E_2, E_3 which together with* 6 *components of the fibre of type I_9 can be contracted to* 3 *singular points of type $\frac{1}{7}(1,3)$.*

Proof. The elliptic fibration $|F_Y|$ on Y induces an elliptic fibration of Y', as the six exceptional curves are contained in fibres. The image $\sigma(E)$ of the -3-curve E is a smooth rational curve not passing through any of the three singular points of Y', and hence splits in X to give three smooth rational curves E_1, E_2, E_3. This implies that the fibres of Y' do not split in X. The fibre containing one of the singular points of Y' gives a fibre of type I_1, the fibre of type I_3 gives a fibre of type I_9, and the multiple fibres give multiple fibres of the same multiplicities. Adding up the Euler number of fibres, we have $c_2(X) = 12$. Thus X is minimal. This proves the first assertion.

Note that the three curves $\sigma(E), \sigma(A_{41}), \sigma(A_{42})$ on Y' form a configuration of smooth rational curves

$$(-3) — (-2) — (-2)$$

which may be contracted to a singular point of type $\frac{1}{7}(1,3)$. Clearly their preimages in X form a disjoint union of three such configurations. □

3. Construction of a fake projective plane

Let X be the surface from Proposition 2.1. Denote by $|F|$ the elliptic fibration on X induced from $|F_Y|$ on Y. Denote clockwise the components of the fibre of $|F|$ of type I_9 by

$$A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2, C_3$$

in such a way that

$$E_1 \cdot A_2 = E_2 \cdot B_2 = E_3 \cdot C_2 = 1.$$

Thus the nine curves

$$A_1, A_2, E_1, \quad B_1, B_2, E_2, \quad C_1, C_2, E_3$$

form a dual diagram

$$(-2)—(-2)—(-3) \quad (-2)—(-2)—(-3) \quad (-2)—(-2)—(-3)$$

which can be contracted to three singular points of type $\frac{1}{7}(1, 3)$. Let

$$\nu: X \to X'$$

be the birational contraction morphism. Let

$$\mathcal{R} \subset H^2(X, \mathbb{Z}) \cong \mathrm{Pic}(X)$$

be the sublattice generated by the classes of the nine exceptional curves, and let $\overline{\mathcal{R}}$ and $\mathcal{R}^{\perp}$ be its primitive closure and its orthogonal complement in $H^2(X, \mathbb{Z})$, respectively. Since $H^2(X, \mathbb{Z})$ is unimodular of rank 10, we see that rank $\mathcal{R}^{\perp} = 1$ and hence

$$l(\mathrm{disc}(\overline{\mathcal{R}})) = l(\mathrm{disc}(\mathcal{R}^{\perp})) = 1.$$

Since $\mathrm{disc}(\mathcal{R})$ is 7-elementary of length 3, we see that $\overline{\mathcal{R}}$ is an overlattice of index 7 of $\mathcal{R}$. This implies that there is a cyclic cover of degree 7

$$\pi: Z \to X'$$

branched exactly at the three singular points of X'. Then Z is a nonsingular surface.

Theorem 3.1. *The surface Z is a fake projective plane, i.e. a surface of general type with $p_g(Z) = 0$ and $c_2(Z) = 3$.*

First we show that Z is of general type.

Lemma 3.2. *The surface Z is of general type.*

Proof. Let X^0 be the smooth part of X', and $\kappa(X^0)$ be the Kodaira dimension of X^0 (that is, its logarithmic Kodaira dimension). Then

$$\kappa(Z) \geq \kappa(X^0) \geq \kappa(X) = 1.$$

We know that X' has Picard number 1, and hence is relatively minimal, i.e. contains no curve C with $C \cdot K_{X'} < 0$ and $C^2 < 0$.

Suppose $\kappa(X^0) = 1$. Then there is an elliptic fibration on X' ([4, Theorem 2.3], [7, Ch.II, Theorem 6.1.4], [5, Theorem 4.1]). This implies that X admits an elliptic fibration whose fibres contain the nine exceptional curves, but no elliptic fibration on X can contain a -3-curve in its fibres because X is minimal. Thus $\kappa(X^0) = 2$ and the assertion follows. □

Lemma 3.3. $c_2(Z) = 3$.

Proof. Let $e(X^0)$ be the Euler number of X^0. Then

$$c_2(Z) = 7e(X^0) + 3 = 7\{e(X) - 12\} + 3 = 3. \quad \square$$

Let

$$E_1 \cdot A_3 = \alpha, \quad E_1 \cdot B_3 = \beta, \quad E_1 \cdot C_3 = \gamma.$$

Then

$$\alpha + \beta + \gamma = 5. \tag{3.1}$$

We claim that there are two possible cases for (α, β, γ);

Case I: $(\alpha, \beta, \gamma) = (2, 1, 2)$.

Case II: $(\alpha, \beta, \gamma) = (1, 3, 1)$.

Let us prove the claim. Since X admits an order 3 automorphism which rotates the I_9 fibre and the 6-sections E_1, E_2, E_3, we have

$$E_2 \cdot (A_3, B_3, C_3) = (\gamma, \alpha, \beta), \quad E_3 \cdot (A_3, B_3, C_3) = (\beta, \gamma, \alpha).$$

The divisor class $E_2 - E_1$ is orthogonal to the class F of a fibre of the elliptic fibration on X and hence can be written in the form

$$E_2 - E_1 = \sum_{i=1}^{3} a_i A_i + \sum_{i=1}^{3} b_i B_i + \sum_{i=1}^{3} c_i C_i$$

for some rational numbers a_i, b_i, c_i. Applying the order 3 automorphism, we get

$$E_3 - E_2 = \sum_{i=1}^{3} a_i B_i + \sum_{i=1}^{3} b_i C_i + \sum_{i=1}^{3} c_i A_i,$$

$$E_1 - E_3 = \sum_{i=1}^{3} a_i C_i + \sum_{i=1}^{3} b_i A_i + \sum_{i=1}^{3} c_i B_i.$$

Adding the three equations side by side, we get

$$a_i + b_i + c_i = 0, \quad i = 1, 2, 3. \tag{3.2}$$

Intersecting $E_2 - E_1$ with A_i, B_i, C_i, E_i, we get 12 equations in nine unknowns a_i, b_i, c_i. The system of these 12 equations together with the three equations from (3.2) has a solution if and only if $(\alpha, \beta, \gamma) = (2, 1, 2)$ or $(1, 3, 1)$. This completes the proof of the claim.

By a direct calculation, we see that the discriminant group of $\mathcal{R}$ is generated by the three elements

$$\frac{1}{7}(A_1 + 2A_2 + 3E_1), \quad \frac{1}{7}(B_1 + 2B_2 + 3E_2), \quad \frac{1}{7}(C_1 + 2C_2 + 3E_3)$$

and hence its discriminant form is isomorphic to $(-3/7)^{\oplus 3}$. Thus a generator of the quotient group $\overline{\mathcal{R}}/\mathcal{R}$ is of the form

$$v = \frac{1}{7}(A_1 + 2A_2 + 3E_1) + \frac{a}{7}(B_1 + 2B_2 + 3E_2) + \frac{b}{7}(C_1 + 2C_2 + 3E_3).$$

Since the intersection numbers $v \cdot A_3$ and $v \cdot B_3$ are integers, we have $a \equiv 4 \pmod 7$ and $b \equiv 2 \pmod 7$ in Case I, and $a \equiv 2 \pmod 7$ and $b \equiv 4 \pmod 7$ in Case II. This determines v uniquely modulo $\mathcal{R}$ in each case. We fix an effective divisor

$$B = \begin{cases} A_1 + 2A_2 + 3E_1 + 4B_1 + B_2 + 5E_2 + 2C_1 + 4C_2 + 6E_3 \text{ (Case I)} \\ A_1 + 2A_2 + 3E_1 + 2B_1 + 4B_2 + 6E_2 + 4C_1 + C_2 + 5E_3 \text{ (Case II).} \end{cases}$$

Then B is divisible by 7 in $\mathrm{Pic}(X)$, so we can write

$$\mathcal{O}_X(B) \cong \mathcal{O}_X(7L)$$

for some divisor L on X. Since X is simply connected, $\mathrm{Pic}(X)$ has no torsion and L is determined uniquely by B up to linear equivalence. We denote by $\mathbb{L}$ the total space of $\mathcal{O}_X(L)$ and by

$$p\colon \mathbb{L} \to X$$

the bundle projection. If $t \in H^0(\mathbb{L}, p^*\mathcal{O}_X(L))$ is the tautological section, and if $s \in H^0(X, \mathcal{O}_X(B))$ is the section vanishing exactly along B, then the zero divisor of $p^*s - t^7$ defines an analytic subspace W in $\mathbb{L}$,

$$W := (p^*s - t^7 = 0) \subset \mathbb{L}.$$

Since B is not reduced, W is not normal.

Lemma 3.4. $H^2(W, \mathcal{O}_W) = 0$.

Proof. Let

$$p\colon W \to X$$

be the restriction to W of the bundle projection $p\colon \mathbb{L} \to X$. Since it is a finite morphism,

$$H^2(W, \mathcal{O}_W) = H^2(X, p_*\mathcal{O}_W).$$

We know that

$$p_*\mathcal{O}_W = \mathcal{O}_X \oplus \mathcal{O}_X(-L) \oplus \mathcal{O}_X(-2L) \oplus \cdots \oplus \mathcal{O}_X(-6L).$$

Thus

$$H^2(W, \mathcal{O}_W) = \bigoplus_{i=0}^{6} H^0(X, \mathcal{O}_X(K_X + iL)).$$

We know $H^0(X, \mathcal{O}_X(K_X)) = 0$.

Assume $K_X + iL$ $(1 \le i \le 6)$ is effective.

Case I: $E_1 \cdot (A_3, B_3, C_3) = (2, 1, 2)$.

To get a contradiction we use the following intersection numbers:

$$\begin{aligned} &L \cdot A_1 = L \cdot A_2 = 0, \qquad L \cdot A_3 = 4, \\ &L \cdot B_1 = -1, \quad L \cdot B_2 = 1, \quad L \cdot B_3 = 4, \\ &L \cdot C_1 = L \cdot C_2 = 0, \qquad L \cdot C_3 = 4, \\ &L \cdot E_1 = -1, \qquad L \cdot E_2 = L \cdot E_3 = -2, \\ &L \cdot K_X = 2, \qquad L \cdot F = 12. \end{aligned}$$

We claim that there are nonnegative integers i_1, i_2, i_3 with $i_1 + i_2 + i_3 = 2i + 1$ and an effective divisor G_i with support contained in the fibre of type I_9 such that the divisor

$$D_i = K_X + iL - i_1E_1 - i_2E_2 - i_3E_3 - G_i$$

is effective. This contradicts the fact that F is represented by an irreducible curve with self-intersection 0, as we have

$$D_i \cdot F = iL \cdot F - 6(i_1 + i_2 + i_3) = 12i - 6(2i + 1) = -6 < 0.$$

This proves that $H^0(X, \mathcal{O}_X(K_X + iL)) = 0$.

It remains to prove the claim.

Assume $i = 6$.

The divisor $D'_1 = K_X + 6L - E_1 - E_2 - E_3$ is effective, because

$$(K_X + 6L) \cdot E_1 = -5, \quad (K_X + 6L) \cdot E_2 = -11, \quad (K_X + 6L) \cdot E_3 = -11.$$

Since $D'_1 \cdot E_3 < 0$, $D'_1 - E_3 = K_X + 6L - E_1 - E_2 - 2E_3$ is effective. Iterating this process, we see that the divisor

$$D_6 = K_X + 6L - 3E_1 - 5E_2 - 5E_3 - (A_1 + 2A_2 + A_3 + 4B_1 + B_2 + C_1 + 3C_2)$$

is effective.

The other cases $i = 5, 4, 3, 2, 1$ can be handled similarly. We give D_i in each case for the sake of completeness:

$$\begin{aligned}
D_5 &= K_X + 5L - 3E_1 - 4E_2 - 4E_3 - (A_1 + 2A_2 + A_3 + 3B_1 + B_2 + C_1 + 2C_2),\\
D_4 &= K_X + 4L - 2E_1 - 3E_2 - 4E_3 - (A_1 + A_2 + 2B_1 + B_2 + C_1 + 2C_2),\\
D_3 &= K_X + 3L - E_1 - 3E_2 - 3E_3 - (A_1 + A_2 + 2B_1 + B_2 + B_3 + C_1 + 2C_2),\\
D_2 &= K_X + 2L - E_1 - E_2 - 3E_3 - (A_1 + A_2 + B_1 + C_1 + 2C_2 + C_3),\\
D_1 &= K_X + L - E_1 - E_2 - E_3 - (B_1 + B_2 + B_3 + C_1 + C_2).
\end{aligned}$$

Case II: $E_1 \cdot (A_3, B_3, C_3) = (1, 3, 1)$.

In this case we use the following intersection numbers:

$$\begin{aligned}
&L \cdot A_1 = L \cdot A_2 = 0, \quad L \cdot A_3 = 4,\\
&L \cdot B_1 = L \cdot B_2 = 0, \quad L \cdot B_3 = 4,\\
&L \cdot C_1 = -1, \quad L \cdot C_2 = 1, \quad L \cdot C_3 = 4,\\
&L \cdot E_1 = -1, \quad L \cdot E_2 = L \cdot E_3 = -2,\\
&L \cdot K_X = 2, \quad L \cdot F = 12.
\end{aligned}$$

The rest of the proof is similar to the previous case. $\square$

Let

$$\mu: W' \to W$$

be the normalization, and

$$f: \widetilde{W} \to W'$$

the minimal resolution. Then $\widetilde{W}$ is a nonsingular surface birational to Z. In particular,

$$p_g(Z) = p_g(\widetilde{W}).$$

Lemma 3.5. $p_g(Z) = 0$.

Proof. Consider the exact sequence of sheaves on W

$$0 \to \mathcal{O}_W \to \mu_* \mathcal{O}_{W'} \to \mathcal{F} \to 0.$$

Since $\mathcal{F}$ is supported in dimension 1,

$$H^2(W, \mathcal{F}) = 0.$$

By Lemma 3.4,

$$H^2(W, \mathcal{O}_W) = 0.$$

Thus the long exact sequence gives

$$H^2(W, \mu_* \mathcal{O}_{W'}) = 0.$$

Since normalization is an affine morphism, this implies that

$$H^2(W', \mathcal{O}_{W'}) = H^2(W, \mu_* \mathcal{O}_{W'}) = 0.$$

Since W' is normal,

$$f_* \mathcal{O}_{\widetilde{W}} = \mathcal{O}_{W'},$$

and hence

$$H^2(W', f_* \mathcal{O}_{\widetilde{W}}) = H^2(W', \mathcal{O}_{W'}) = 0.$$

Note that the direct image sheaf $R^1 f_* \mathcal{O}_{\widetilde{W}}$ is supported in dimension 0. Thus

$$H^1(W', R^1 f_* \mathcal{O}_{\widetilde{W}}) = 0.$$

Now the Leray spectral sequence gives

$$p_g(\widetilde{W}) = H^2(\widetilde{W}, \mathcal{O}_{\widetilde{W}}) = 0. \quad \square$$

Now Theorem 3.1 follows from Lemmas 3.2, 3.3 and 3.5.

Acknowledgements

I would like to thank I. Dolgachev for many useful conversations. I am also grateful to F. Kato for an e-mail exchange. This research was supported by KOSEF grant R01-2003-000-11634-0.

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